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SHAPE-PRESERVING C^2 CUBIC POLYNOMIAL INTERPOLATING SPLINES

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ABSTRACT. In this paper we propose a method to construct shape-preserving C^2 cubic polynomial splines interpolating convex and/or monotonic data. For such given data, the existence or nonexistence of such interpolating splines can be expressed in terms of existence or nonexistence of solutions for a system of linear inequalities in two unknowns.

0. INTRODUCTION

In many interpolation problems it is important that the solution preserves some shape properties such as convexity or monotonicity. Classical methods (the polynomial spline functions being the most widely used) usually ignore these kinds of conditions and thus yield solutions exhibiting undesirable inflections or oscillations. This is the reason why many investigations during the last years have been directed towards interpolation by means of shape-preserving polynomial spline functions.

In [6] McAllister and Roulier, and in [13] Schumaker, have studied quadratic splines which preserve monotonicity and convexity. In [5] Fritsch and Carlson have studied cubic splines that preserve monotonicity. In [1, 2] Costantini and Morandi have studied cubic splines which preserve both convexity and monotonicity. All of these splines are C^1.

Other authors (Neuman [10, 11] and Mettke [9]) have imposed additional conditions on the monotone, convex data, which yield a solution that belongs to a subspace of polynomial splines. Moreover, McAllister and Roulier [7], and Passow and Roulier [12] have shown that it may be impossible to construct monotonic and convex splines of given degree and deficiency. In [8] Medina gives a survey on shape-preserving interpolation by means of polynomial or of other classes of splines.

In this paper we propose a method to construct C^2 cubic polynomial interpolation splines. Functions of this kind, of course, do not always exist for arbitrary convex monotone data sets. For convex increasing data—this, as we

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shall see later, does not limit generality—the existence or nonexistence of such a C^2 cubic polynomial interpolation spline can be expressed in terms of the existence or nonexistence of solutions for a system of linear inequalities in two unknowns.

1. Notation and definitions

Let $J = \{0, 1, \ldots, n-1\}$ and $K = \{1, 2, \ldots, n-1\}$. Suppose $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ is a partition of $[a, b]$, and suppose that $y_0, y_1, \ldots, y_n$ are $n+1$ real numbers. Let $I_i = [x_i, x_{i+1}]$ for $i \in J$, $\Delta_i = (y_{i+1} - y_i)/(x_{i+1} - x_i)$ for $i \in J$, $h_i = x_{i+1} - x_i$ for $i \in J$, and $x_{i+1/3} = x_i + h_i/3$, $x_{i+2/3} = x_i + 2h_i/3$ for $i \in J$.

Given the set of points M_i, with $M_i = (x_i, y_i)$ for $i = 0, 1, \ldots, n$, we say that:

(a) the data are increasing (resp. decreasing) if and only if

$$y_0 \le y_1 \le y_2 \le \cdots \le y_{n-1} \le y_n \quad (\text{resp. } y_0 \ge y_1 \ge \cdots \ge y_n);$$

(b) the data are convex (resp. concave) if and only if

$$\Delta_0 \le \Delta_1 \le \cdots \le \Delta_{n-2} \le \Delta_{n-1} \quad (\text{resp. } \Delta_0 \ge \Delta_1 \ge \cdots \ge \Delta_{n-1}).$$

2. The problem

For clarity we shall assume that the data are convex and increasing, and we shall try to find a degree-3 polynomial spline function, denoted by s, of class $C^2[a, b]$, interpolating the points M_i, $i = 0, 1, \ldots, n$, and preserving the shape of the data.

These conditions are expressed as follows:

(i) for $i \in J$ and $x \in I_i$, $s(x) = P_i(x)$, where P_i is a degree-3 polynomial;
(ii) $s(x_0) = P_0(x_0) = y_0$; for $i \in K$, $s(x_i) = P_{i-1}(x_i) = P_i(x_i) = y_i$; $s(x_n) = P_{n-1}(x_n) = y_n$;
(iii) for $i \in K$, $P'_{i-1}(x_i) = P'_i(x_i)$, $P''_{i-1}(x_i) = P''_i(x_i)$;
(iv) for $i \in J$ and $x \in I_i$, $P'_i(x) \ge 0$ and $P''_i(x) \ge 0$.

In §3 a solution satisfying conditions (i)–(iv) is constructed. The construction is simplified when the data are only increasing (resp. only convex), since in this case, in (iv), only the condition of positivity of the first (resp. the second) derivative is required. Of course, by symmetry the case of concave and decreasing (resp. only decreasing or only concave) data is treated in the same way. One simply changes the sign in (iv).

3. The proposed solution

3.1. Let $y_{i+1/3}, y_{i+2/3}$, $i \in J$, be $2n$ real numbers, and

$$P_i(x) = \frac{1}{(h_i)^3}\{y_i(x_{i+1} - x)^3 + 3y_{i+1/3}(x_{i+1} - x)^2(x - x_i) + 3y_{i+2/3}(x_{i+1} - x)(x - x_i)^2 + y_{i+1}(x - x_i)^3\} \quad \text{for } x \in [x_i, x_{i+1}].$$

By construction, s satisfies (i) and (ii). For further use, we give the following first and second derivatives:

$$P_i'(x) = \frac{3}{h_i^3}\{(y_{i+1/3} - y_i)(x_{i+1} - x)^2 + 2(y_{i+2/3} - y_{i+1/3})(x_{i+1} - x)(x - x_i) + (y_{i+1} - y_{i+2/3})(x - x_i)^2\},$$

$$P_i''(x) = \frac{6}{(h_i)^3}\{(y_{i+2/3} - 2y_{i+1/3} + y_i)(x_{i+1} - x) + (y_{i+1} - 2y_{i+2/3} + y_{i+1/3})(x - x_i)\}.$$

Theorem 1. *The function* s, *defined by*

$$s(x) = P_i(x) \quad \textit{for } i \in J \textit{ and } x \in I_i,$$

satisfies (iv) *if and only if the following conditions hold*:

$$(1) \qquad y_i \le y_{i+1/3} \le y_{i+2/3} \le y_{i+1}, \qquad i \in J,$$

$$(2) \qquad \frac{y_{i+1/3} - y_i}{x_{i+1/3} - x_i} \le \frac{y_{i+2/3} - y_{i+1/3}}{x_{i+2/3} - x_{i+1/3}} \le \frac{y_{i+1} - y_{i+2/3}}{x_{i+1} - x_{i+2/3}}, \qquad i \in J.$$

Proof. Suppose that conditions (1) and (2) hold. Obviously, $P_i(x_i) = y_i$ and $P_i(x_{i+1}) = y_{i+1}$. It is furthermore known (Davis [3, pp. 114–115]), or can easily be verified using the expressions of P_i' and P_i'' given in 3.3, that P_i satisfies the following properties: for all $x \in I_i$, $P_i'(x) \ge 0$ (by (1)) and $P_i''(x) \ge 0$ (by (2)).

Conversely, assume P_i convex and increasing for $i \in J$. This implies, in particular, that the first and second derivatives of P_i at x_i and x_{i+1} are positive for $i \in J$. Conditions $P_i''(x_i) \ge 0$ for $i \in J$, and $P_i''(x_{i+1}) \ge 0$ for $i \in J$ read:

$$(\alpha) \quad y_{i+2/3} - 2y_{i+1/3} + y_i \ge 0 \quad \text{and} \quad y_{i+1} - 2y_{i+2/3} + y_{i+1/3} \ge 0 \quad \text{for } i \in J.$$

Similarly, conditions $P_i'(x_i) \ge 0$ for $i \in J$, and $P_i'(x_{i+1}) \ge 0$ for $i \in J$ read:

$$(\beta) \qquad y_{i+1/3} - y_i \ge 0 \quad \text{and} \quad y_{i+1} - y_{i+2/3} \ge 0 \quad \text{for } i \in J.$$

One can easily check that conditions (α) and (β) imply (1) and (2). □

Thus, solving our problem is equivalent to finding $2n$ real numbers $y_{i+1/3}$, $y_{i+2/3}$ satisfying (1) and (2) and ensuring, in addition, the continuity of the first and second derivatives at the nodes x_i, $i \in K$.

3.2. To simplify, set for $i \in J$,

$$d_i = \frac{y_{i+1/3} - y_i}{x_{i+1/3} - x_i} = 3\frac{y_{i+1/3} - y_i}{h_i},$$

$$d_{i+1/3} = \frac{y_{i+2/3} - y_{i+1/3}}{x_{i+2/3} - x_{i+1/3}} = 3\frac{y_{i+2/3} - y_{i+1/3}}{h_i},$$

$$d_{i+2/3} = \frac{y_{i+1} - y_{i+2/3}}{x_{i+1} - x_{i+2/3}} = 3\frac{y_{i+1} - y_{i+2/3}}{h_i}.$$

Then we have

$$d_i + d_{i+1/3} + d_{i+2/3} = 3\Delta_i, \qquad i \in J. \tag{3}$$

We now rewrite conditions (1) and (2), using d_i, $d_{i+1/3}$, $d_{i+2/3}$, $i \in J$. Then (1) is equivalent to the following system of $3n$ inequalities in $3n$ unknowns:

$$d_i \geq 0, \quad d_{i+1/3} \geq 0, \quad d_{i+2/3} \geq 0 \quad \text{for } i \in J, \tag{1$'$}$$

and (2) is equivalent to the following system of $2n$ inequalities in $3n$ unknowns:

$$d_i \leq d_{i+1/3} \leq d_{i+2/3} \quad \text{for } i \in J. \tag{2$'$}$$

Thus, our problem is equivalent to determining $\{d_i, d_{i+1/3}, d_{i+2/3}\}$, $i \in J$, satisfying $(1)'$, $(2)'$, and (3) in such a manner that the continuity of the first and second derivatives at the nodes is ensured.

3.3. From the expressions of P_i' and P_i'' (see 3.1) we obtain

$$P_i'(x_i) = d_i, \qquad P_i'(x_{i+1}) = d_{i+2/3},$$

$$P_i''(x_i) = \frac{2}{h_i}(d_{i+1/3} - d_i), \qquad P_i''(x_{i+1}) = \frac{2}{h_i}(d_{i+2/3} - d_{i+1/3}).$$

So, to satisfy the continuity of the first derivative at the nodes, it is necessary and sufficient to have

$$d_i = d_{i-1+2/3} \quad \text{for } i \in K. \tag{4}$$

Similarly, to satisfy the continuity of the second derivative at the nodes, it is necessary and sufficient to have

$$\frac{d_{i-1+2/3} - d_{i-1+1/3}}{h_{i-1}} = \frac{d_{i+1/3} - d_i}{h_i} \quad \text{for } i \in K. \tag{5}$$

In view of (4), it is natural to set $d_n = d_{n-1+2/3}$.

We have obtained the following theorem.

Theorem 2. *To solve the problem of §2, it is necessary and sufficient to determine* $\{d_i, d_{i+1/3}, d_{i+2/3}\}$, $i \in J$, *satisfying the five conditions* $(1)'$, $(2)'$, (3), (4), *and* (5).

These conditions lead to the following system of linear equations and inequalities in $3n$ unknowns:

$$(\mathrm{S}) \begin{cases} d_i \geq 0, \; d_{i+1/3} \geq 0, \; d_{i+2/3} \geq 0 & \text{for } i \in J, \quad (1)' \\ d_i \leq d_{i+1/3} \leq d_{i+2/3} & \text{for } i \in J, \quad (2)' \\ d_i + d_{i+1/3} + d_{i+2/3} = 3\Delta_i & \text{for } i \in J, \quad (3) \\ d_i = d_{i-1+2/3} & \text{for } i \in K, \quad (4) \\ \dfrac{d_{i+1/3} - d_i}{h_i} = \dfrac{d_{i-1+2/3} - d_{i-1+1/3}}{h_{i-1}} & \text{for } i \in K. \quad (5) \end{cases}$$

In what follows we shall denote $h_i h_{i+1}/(h_i + h_{i+1})$ by H_i.

3.4. Let us rewrite this system, modifying $(2)'$, (3), and (5) according to (4):

$$\begin{cases} d_i \geq 0, \; d_{i+1/3} \geq 0, \; d_{i+2/3} \geq 0 & \text{for } i \in J, \quad (1)' \\ d_i \leq d_{i+1/3} \leq d_{i+1} & \text{for } i \in J, \quad (2)'' \\ d_i + d_{i+1/3} + d_{i+1} = 3\Delta_i & \text{for } i \in J, \quad (3)' \\ d_i = d_{i-1+2/3}, & \text{for } i \in K, \quad (4) \\ d_i = H_{i-1}\left(\dfrac{d_{i+1/3}}{h_i} + \dfrac{d_{i-1+1/3}}{h_{i-1}}\right) & \text{for } i \in K. \quad (5)' \end{cases}$$

In view of (4), the inequalities $(1)'$, $(2)''$ together are equivalent to

$$(6) \qquad 0 \leq d_0 \leq d_{0+1/3} \leq d_1 \leq d_{1+1/3} \leq d_2 \leq \cdots \leq d_{n-1} \leq d_{n-1+1/3} \leq d_n.$$

Given that the quantities $d_{i+1/3}$ and $d_{i-1+1/3}$ are positive and, by $(5)'$, d_i is a weighted average of them, d_i is necessarily included between these two numbers, and inequalities (6) can be written more simply as

$$(\mathrm{I}) \qquad 0 \leq d_0 \leq d_{0+1/3} \leq d_{1+1/3} \leq \cdots \leq d_{n-2+1/3} \leq d_{n-1+1/3} \leq d_n.$$

System (S) now reads as follows:

$$\begin{cases} 0 \leq d_0 \leq d_{0+1/3} \leq d_{1+2/3} \leq \cdots \leq d_{n-2+1/3} \leq d_{n-1+1/3} \leq d_n, & (\mathrm{I}) \\ d_i + d_{i+1/3} + d_{i+1} = 3\Delta_i \qquad \text{for } i \in J, & (3)' \\ d_i = d_{i-1+2/3} \qquad \text{for } i \in K, & (4) \\ d_i = H_{i-1}\left(\dfrac{d_{i+1/3}}{h_i} + \dfrac{d_{i-1+1/3}}{h_{i-1}}\right) \qquad \text{for } i \in K. & (5)' \end{cases}$$

Using $(5)'$, we express $(3)'$ in the following way:

$$(\mathrm{E}) \begin{cases} (h_0 + H_0)\dfrac{d_{0+1/3}}{h_0} + H_0\dfrac{d_{1+1/3}}{h_1} = 3\Delta_0 - d_0, \\ H_{i-1}\dfrac{d_{i-1+1/3}}{h_{i-1}} + (h_i + H_{i-1} + H_i)\dfrac{d_{i+1/3}}{h_i} + H_i\dfrac{d_{i+1+1/3}}{h_{i+1}} = 3\Delta_i \\ \qquad\qquad\qquad\qquad \text{for } i = 1, 2, \ldots, n-2, \\ H_{n-2}\dfrac{d_{n-2+1/3}}{h_{n-2}} + (h_{n-1} + H_{n-2})\dfrac{d_{n-1+1/3}}{h_{n-1}} = 3\Delta_{n-1} - d_n. \end{cases}$$

Finally, (S) is equivalent to the system of linear equations and inequalities (I), (E), (4), $(5)'$.

3.5. We now study system (E) of n linear equations in n unknowns $d_{i+1/3}$, $i \in J$, where d_0 and d_n are parameters in the right-hand side. In terms of the unknowns $D_i = d_{i+1/3}/h_i$, the corresponding matrix, denoted by H, can be written as follows:

$$\begin{pmatrix} h_0 + H_0 & H_0 & & & & \\ H_0 & h_1 + H_0 + H_1 & H_1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & H_{n-3} & h_{n-2} + H_{n-3} + H_{n-2} & H_{n-2} \\ & & & H_{n-2} & h_{n-1} + H_{n-2} \end{pmatrix}.$$

This is a diagonally dominant, tridiagonal, symmetric, hence invertible matrix; thus, (E) has a unique solution. The unknowns D_i, $i \in J$, hence $d_{i+1/3}$, will be written as linear combinations of d_0 and d_n. Once $d_{i+1/3}$, $i \in J$, are determined, we obtain d_i and $d_{i-1+2/3}$ for $i \in K$, using $(5)'$ and (4), respectively.

There remains to verify inequalities (I):

$$d_0 \geq 0, \, d_{0+1/3} - d_0 \geq 0, \, d_{1+1/3} - d_{0+1/3} \geq 0, \ldots, \\ d_{n-1+1/3} - d_{n-2+1/3} \geq 0, \, d_n - d_{n-1+1/3} \geq 0.$$

3.6. The system (E) can be written symbolically in the following manner: $H\tilde{d} = \tilde{\Delta} - d_0 e_1 - d_n e_n$, where $\tilde{d}$, $\tilde{\Delta}$, e_1, e_n are respectively the vectors

$$\tilde{d} = (D_0, D_1, \ldots, D_{n-1})^t, \qquad \tilde{\Delta} = (3\Delta_0, 3\Delta_1, \ldots, 3\Delta_{n-1})^t, \\ e_1 = (1, 0, \ldots, 0)^t, \qquad e_n = (0, 0, \ldots, 1)^t.$$

Therefore, $\tilde{d} = H^{-1}\tilde{\Delta} - d_0 H^{-1} e_1 - d_n H^{-1} e_n$.

The proposed method can be described by the following

Algorithm. (a) Express $d_{0+1/3}, d_{1+1/3}, \ldots, d_{n-1+1/3}$ as functions of the two free parameters (d_0, d_n) by solving three systems of linear equations with the same tridiagonal matrix.

(b) Substitute these expressions in (I), which becomes a system of $n+2$ linear inequalities in the two unknowns (d_0, d_n).

(c) Find a (d_0, d_n) in the polygon defined by (I).

(d) IF (I) is empty, THEN our problem has no solution: STOP.

ELSE calculate numerically $d_{0+1/3}, d_{1+1/3}, \ldots, d_{n-1+1/3}$, then P_i for $i \in J$: STOP.

Remarks. (R_1) Because of the uniqueness of the solution of (E), if the interpolation data are points chosen in the graph of a convex monotone polynomial of degree ≤ 3, and if we choose d_0 (resp. d_n) equal to the value of the derivative of the interpolating function at the first node (resp. at the last node), the solution is the function itself.

(R_2) This method can be extended to splines of higher degrees and smoothness (for instance degree 4 and smoothness class C^3). The number of parameters will be larger, thus giving rise to a larger linear system of inequalities.

(R_3) Dupin and Freville [4], for a uniform mesh, give some sufficient conditions for the existence of such shape-preserving C^2 cubic polynomial interpolating splines and give a corresponding algorithm.

(R_4) For solving (I), there exist algorithms requiring a number of iterations which is bounded by a polynomial in terms of the size of the problem (Karmarkar's and Kachyan's algorithm).

Of course, the simplex method, or even a graphical method, can also be used.

3.7. **Uniform subdivision.** In this case, $h_0 = h_1 = h_2 = \cdots = h_{n-2} = h_{n-1}$, and (E) becomes

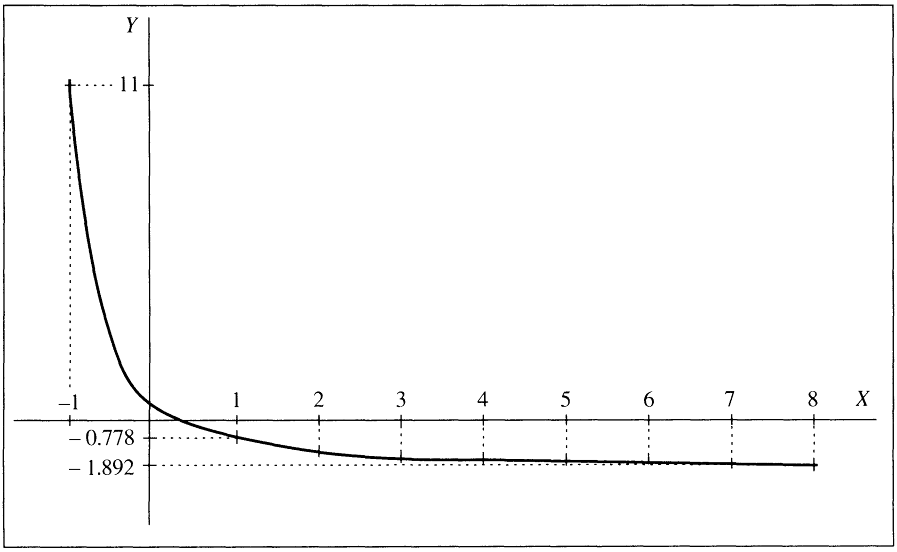
$$\begin{cases} \frac{3}{2}d_{0+1/3} + \frac{1}{2}d_{1+1/3} = 3\Delta_0 - d_0, \\ \frac{1}{2}d_{i-1+1/3} + 2d_{i+1/3} + \frac{1}{2}d_{i+1+1/3} = 3\Delta_i \quad \text{for } i = 1, 2, \ldots, n-2, \\ \frac{1}{2}d_{n-2+1/3} + \frac{3}{2}d_{n-1+1/3} = 3\Delta_{n-1} - d_n. \end{cases}$$

The corresponding matrix H is now also symmetric relative to its center. So $(z_0, z_1, \ldots, z_{n-1})^t = H^{-1}e_n$ is obtained from $(t_0, t_1, \ldots, t_{n-1})^t = H^{-1}e_1$ by $z_i = t_{n-1-i}$ for $i \in J$.

3.8. **Example.** We interpolate ten points $M_i(x_i, z_i)$ of the graph of the function $f(t) = (-9t+2)/(4t+5)$ (cf. [8, p. 50]). This function is decreasing and convex on the interval $[-1, 8]$. We use a uniform subdivision $h_i = 1$ for $i = 0, 1, \ldots, 8$.

t_i	−1	0	1	2	3	4	5	6	7	8
z_i	11	0.4	−0.77777	−1.23077	−1.47059	−1.6190	−1.72	−1.79310	−1.84848	−1.89189

Taking $d_0 = -27$ and $d_9 = -0.03$, we obtain a solution satisfying inequalities (I). Its graph is shown below.



Conclusions

A direct, inexpensive, constructive method for interpolating convex, monotone data with shape-preserving C^2 cubic polynomial splines is proposed. Whenever the corresponding polyhedron in $\mathbf{R}^2$ is nonempty, it determines the two degrees of freedom that occur in the classical cubic spline interpolation problem in such a way as to ensure the shape conditions. The technique seems promising for higher degrees and smoothness and from the point of view of accuracy.

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